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Finite amplitude effect on the stability of a jet of circular cross-section

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The effect of finite amplitude on the stable and unstable states of a column of an ideal fluid of circular cross-section under the action of surface tension is studied. The method of solution is a formal extension of the linearized theory; it consists of assuming that the exact solution may be expanded in a power series of a small parameter characterizing the amplitude. The calculation is carried out to the point where the first non-trivial term of the finite amplitude effect is obtained. For the stable states, the result shows that the characteristic wavelength of a disturbance which appears to be stationary with respect to an observer is decreased by the finite amplitude effect. For the unstable states, it reveals that the growth rate depends not only on the wavelength and the magnitude but also on the type of disturbance imposed initially. The last result is a direct consequence of the fact that two independent types of initial disturbance, the disturbance of the velocity field and the disturbance of the free surface, may be imposed simultaneously on the jet.

1. Introduction

The object of this paper is to study the effect of finite amplitude on the stable and unstable states of a cylindrical column of an ideal fluid, of circular cross-section, under the action of surface tension. A columnar jet of circular cross-section is formed when liquid issues under pressure through a small circular orifice into the air when the gravity effect may be neglected. It is well known that in consequence of surface tension a column of circular jet may be unstable under a small disturbance. The instability causes the jet to disintegrate into drops. This interesting phenomenon was examined and described in the experiments by Savart (1833), Haenlein (1932), Grant & Middleman (1966) and many others.

Based on the linearized theory, Lord Rayleigh (1878, 1879) gave a detailed analytical explanation of this phenomenon. He considered a columnar circular jet of an ideal fluid under a small disturbance which was periodic in the axial direction of the jet. He showed that the jet was always stable except when the disturbance was axisymmetric and had a wavelength longer than the circumference of the jet. He further showed that among all the unstable disturbances there was a most unstable mode which occurred when the wavelength of the disturbance was 1·435 times the circumference of the jet. From this result he concluded that, if the initial magnitude of the disturbance was sufficiently small compared with the radius of the jet, this characteristic length would eventually decide the size of the drops when the jet disintegrated.

In extending Rayleigh's theory, Weber (1931) considered the stability of a viscous jet. His linearized theory indicated that the viscosity would not change the stability criterion as predicted by the inviscid theory. However, the viscous effect would cause the wavelength of the most unstable state to become longer than that predicted by the inviscid theory. He also considered the effect of the surrounding air on the stability of a jet. The influence of the ambient air pressure was found to shorten the break-up time of the jet. But, if the velocity of the jet when issued into the air was not too large, the effect of the surrounding air on the jet might be neglected. His results were qualitatively in agreement with the experimental observations.

Haenlein (1932) investigated experimentally the process of disintegration of cylindrical jets of liquids of different physical properties. Among his experiments, the one most pertinent to the present investigation is the disintegration of a cylindrical jet of water, which had the largest ratio of the surface tension to the viscosity among all the liquids considered by him. He reported that at a moderate jet velocity, when the influence of the surrounding air might be neglected, the water jet disintegrated at wavelengths varying from 1·4 to 2·2 times the circumference of the jet. At higher velocities, his report indicated that the influence of the surrounding air gradually became dominant and the break-up behaviour of the jet became quite different from that when the jet was at lower velocity.

The observed variations in the break-up wavelength of a cylindrical jet indicate that the properties of the initial disturbance may play an important role in the process of disintegration of a jet. To investigate this role a non-linear study of the problem must be made. The present work is devoted to this purpose.

In the present paper the finite amplitude effect on the growth rate of an unstable disturbance as well as on the propagation speed of a stable, axisymmetric disturbance will be studied. Neither the viscous effect nor the influence of the surrounding air will be considered. The method of solution is a formal extension of the linearized theory; it consists in assuming that the velocity potential, the free surface displacement and the growth rate, or the propagation speed, may be expanded in power series of a small parameter ϵ characterizing the amplitude. By requiring that these formal expansions satisfy the exact governing equations and the boundary conditions for all values of ϵ, sets of linearized boundary-value problems can be obtained. The calculations presented in this paper are carried up to the third-order terms in ϵ so that the first non-trivial term of the finite amplitude effect on the growth rate, or on the propagation speed, may be obtained. This perturbation technique has been applied by many authors to problems of water waves with finite amplitude (see Wehausen & Laitone (1960); for more recent works, see Tadjbakhsh & Keller (1960), Verma & Keller (1962) and Concus (1962)), to non-linear stability problems (see Segel 1965), and by Keller &

Ting (1966) to periodic vibrations of systems governed by non-linear partial differential equations.

The formulation and the solutions of the present problem are given in §§2–4. The results are also discussed.

2. Formulation of the problem

Consider an infinitely long, columnar jet of an ideal fluid of circular cross-section standing still in an inertial frame of reference (r, θ, z) as shown in figure 1. We assume that the jet is under the action of surface tension only. The problem is formulated non-dimensionally with the radius of the undisturbed jet, a, as the characteristic length and $(T/\rho a)^{\frac{1}{2}}$ as the characteristic velocity, where T is the surface tension and ρ the density of the fluid.

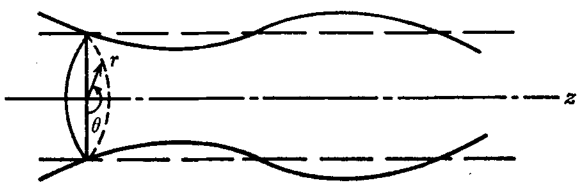

FIGURE 1. A cylindrical jet of circular cross-section and its co-ordinate axes.

To study the stability of this state a small disturbance, periodic in the z-direction, is assumed to be applied to the jet. The resulting flow is assumed to remain irrotational so that a perturbation velocity potential $\phi(r, \theta, z, t)$, which is periodic in the z-direction, exists and satisfies the Laplace equation

$$\Delta\phi(r, \theta, z, t) = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz} = 0. \tag{1}$$

Let us denote $\zeta(\theta, z, t)$ to be the displacement in the r-direction of the free surface of the jet from the originally undisturbed one; then the equation for the disturbed free surface may be written as

$$r = 1 + \zeta(\theta, z, t). \tag{2}$$

We note that $\zeta(\theta, z, t)$ is assumed to have a period λ in the z-direction. Due to the conservation of mass and the periodicity of the disturbance, we should have the relation

$$\frac{1}{2}\int_0^{2\pi/k} dz \int_0^{2\pi} (1+\zeta)^2 d\theta = \pi(2\pi/k), \tag{3}$$

where $k = 2\pi/\lambda$.

The boundary conditions on the free surface of the jet are the kinematic and the dynamic boundary conditions, which are (cf. Lamb 1932, pp. 455–456)

$$\phi_r = \phi_z \zeta_z + \frac{1}{r^2}\phi_\theta \zeta_\theta + \zeta_t, \quad \text{on} \quad r = 1 + \zeta(\theta, z, t), \tag{4}$$

and

$$\phi_t+\frac{1}{2}\left[\phi_r^2+\left(\frac{\phi_\theta}{1+\zeta}\right)^2+\phi_z^2\right]+\frac{1}{[1+(\zeta_\theta/\{1+\zeta\})^2+\zeta_z^2]^{\frac{3}{2}}}\left\{\frac{1}{1+\zeta}\left[1+2\left(\frac{\zeta_\theta}{1+\zeta}\right)^2+\zeta_z^2\right]\right.$$
$$\left.-\frac{1+\zeta_z^2}{(1+\zeta)^2}\zeta_{\theta\theta}-\left[1+\left(\frac{\zeta_\theta}{1+\zeta}\right)^2\right]\zeta_{zz}+\frac{2\zeta_\theta\zeta_z\zeta_{\theta z}}{(1+\zeta)^2}\right\}=1, \quad \text{on} \quad r=1+\zeta(\theta,z,t), \tag{5}$$

respectively. It should be mentioned that a function of t has been temporarily deleted from the dynamic boundary condition (5). This function of t can be determined later when the physical problem is completely specified.

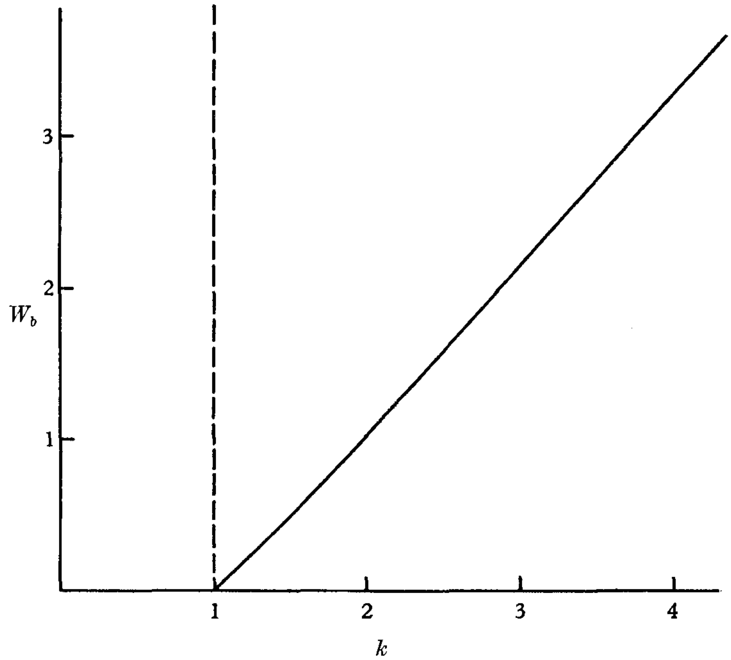

FIGURE 2. Linearized stationary solutions for an axisymmetric disturbance.

If we regard both ϕ and ζ as infinitesimal quantities and neglect the non-linear terms in equations (1), (3), (4) and (5), a linearized solution can be obtained and is given by

$$\phi(r,\theta,z,t) = \mathrm{Re}\,(a\exp\{\sigma t+ikz\})\,I_m(kr)\cos m\theta, \tag{6}$$

where a is an arbitrary constant, I_m is the Bessel function of order m, with imaginary argument, m may be any integer, including zero, and

$$\sigma = \pm\left[\frac{I_m'(k)}{I_m(k)}k(1-m^2-k^2)\right]^{\frac{1}{2}}. \tag{7}$$

In (7), the notation $I_m'(k)$ is used to indicate $dI_m(k)/dk$. We see from (7) that instability may occur only when $m = 0$. For $m = 0$ and $k < 1$ the quantity under the square-root sign in (7) is always positive and has a maximum at $k = 0{\cdot}697$. This value of k gives us the most unstable mode. The above solution and its explanation were given by Rayleigh (1878, 1879).

When the quantity under the square-root sign in (7) is negative, which gives us stable modes, we may rewrite (7) in the form

$$\sigma = \pm ik\left[\frac{I'_m(k)\,(m^2+k^2-1)}{I_m(k)\,k}\right]^{\frac{1}{2}}. \tag{8}$$

The phase velocity c of the stable modes is then

$$c = \left[\frac{I'_m(k)\,(m^2+k^2-1)}{I_m(k)\,k}\right]^{\frac{1}{2}}. \tag{9}$$

To an observer moving with a velocity $U/(T/\rho a)^{\frac{1}{2}}$ in the z-direction with respect to the jet, a stable mode appears to be stationary if

$$W_b = c^2 = \frac{I'_m(k)\,(m^2+k^2-1)}{I_m(k)\,k}, \tag{10}$$

where $W_b = U^2/(T/\rho a)$ is the so-called Weber number.

Equation (10) is plotted in figure 2 for the case $m = 0$. In that figure, the ordinate denotes the Weber number W_b and the abscissa denotes k, which, when the dimensions are restored, represents the ratio of the circumference of the originally undisturbed jet to the wavelength of the periodic disturbance.

3. Finite amplitude effect on the stable states

To study the effect of finite amplitude on the stable states, it is sufficient to study only those disturbances which appear to be stationary to an observer. In the following analysis only the axisymmetric case, or $m = 0$, is considered.

From the linearized solution, it is seen that to an observer having a relative velocity $U/(T/\rho a)^{\frac{1}{2}}$ with respect to the jet a disturbance appears to be stationary if

$$W_b = \frac{U^2\rho a}{T} = \frac{I_1(k)\,(k^2-1)}{I_0(k)\,k}, \tag{11}$$

where $I'_0(k) = I_1(k)$ has been used. However, if the effect of amplitude is also considered, it is expected that the Weber number should not only be a function of the wave-number k but should also depend on the amplitude. To obtain such a functional relationship, the following procedures will be used.

For this problem, it is more convenient to transform the free surface boundary conditions to a frame of reference moving with the observer so that the problem becomes steady. In doing so, the final forms of the free surface boundary conditions are

$$\phi_r = (\phi_z - W_b^{\frac{1}{2}})\,\zeta_z, \quad \text{on} \quad r = 1+\zeta(z), \tag{12}$$

and

$$-W_b^{\frac{1}{2}}\phi_z + \tfrac{1}{2}(\phi_r^2+\phi_z^2) + \frac{1}{(1+\zeta_z^2)^{\frac{3}{2}}}\left[\frac{1+\zeta_z^2}{1+\zeta} - \zeta_{zz}\right] = 1, \quad \text{on} \quad r = 1+\zeta(z). \tag{13}$$

In addition to (12) and (13), we also have, from (1) and (3),

$$\phi_{rr} + \tfrac{1}{2}\phi_r + \phi_{zz} = 0, \qquad \text{for} \quad r \leqslant 1+\zeta(z), \quad -\infty < z < \infty, \tag{14}$$

and

$$\int_0^{2\pi/k} (1+\zeta)^2\,dz = 2\pi/k. \tag{15}$$

We now assume that the amplitude of the disturbances is small and that the Weber number may be expanded in a power series of a small parameter ϵ characterizing the amplitude; we may write

$$W_b^{\frac{1}{2}} = \alpha_0(k) + \epsilon\dot{\alpha}(k) + \epsilon^2\ddot{\alpha}(k) + O(\epsilon^3), \tag{16}$$

where $\alpha_0(k)$, $\dot{\alpha}(k)$ and $\ddot{\alpha}(k)$ are functions of k only. Similarly, we assume that both the perturbation potential ϕ and the free surface displacement ζ may be expanded in power series of ϵ as shown in the following:

$$\phi(r,z) = \epsilon\dot{\phi}(r,z) + \epsilon^2\ddot{\phi}(r,z) + \epsilon^3\dddot{\phi}(r,z) + O(\epsilon^4), \tag{17}$$

$$\zeta(z) = \epsilon\dot{\zeta}(z) + \epsilon^2\ddot{\zeta}(z) + \epsilon^3\dddot{\zeta}(z) + O(\epsilon^4). \tag{18}$$

Substituting (16), (17) and (18) into (12), (13), (14) and (15) and equating, in the resulting equations, the coefficients of like powers of ϵ yield the following sets of equations:

$$\dot{\phi}_{rr} + \frac{1}{r}\dot{\phi}_r + \dot{\phi}_{zz} = 0, \qquad \text{for} \quad r \leqslant 1, \quad -\infty < z < \infty, \tag{19}$$

$$\int_0^{2\pi/k} \dot{\zeta}\,dz = 0, \tag{20}$$

$$\dot{\phi}_r + \alpha_0\dot{\zeta}_z = 0, \quad \text{on} \quad r = 1, \tag{21}$$

$$\alpha_0\dot{\phi}_z + \dot{\zeta}_{zz} + \dot{\zeta} = 0, \quad \text{on} \quad r = 1; \tag{22}$$

$$\ddot{\phi}_{rr} + \frac{1}{r}\ddot{\phi}_r + \ddot{\phi}_{zz} = 0, \qquad \text{for} \quad r \leqslant 1, \quad -\infty < z < \infty, \tag{23}$$

$$\int_0^{2\pi/k} (2\ddot{\zeta} + \dot{\zeta}^2)\,dz = 0, \tag{24}$$

$$\ddot{\phi}_r + \alpha_0\ddot{\zeta}_z + \dot{\alpha}\dot{\zeta}_z = \dot{\phi}_z\dot{\zeta}_z - \dot{\phi}_{rr}\dot{\zeta}, \quad \text{on} \quad r = 1, \tag{25}$$

$$\ddot{\zeta}_{zz} + \ddot{\zeta} + \alpha_0\ddot{\phi}_z + \dot{\alpha}\dot{\phi}_z = -\alpha_0\dot{\phi}_{zr}\dot{\zeta} + \tfrac{1}{2}(\dot{\phi}_r^2 + \dot{\phi}_z^2 - \dot{\zeta}_z^2) + \dot{\zeta}^2, \quad \text{on} \quad r = 1; \tag{26}$$

and

$$\dddot{\phi}_{rr} + \frac{1}{r}\dddot{\phi}_r + \dddot{\phi}_{zz} = 0, \qquad \text{for} \quad r \leqslant 1, \quad -\infty < z < \infty, \tag{27}$$

$$\int_0^{2\pi/k} (\dddot{\zeta} + \dot{\zeta}\ddot{\zeta})\,dz = 0, \tag{28}$$

$$\dddot{\phi}_r + \alpha_0\dddot{\zeta}_z + \ddot{\alpha}\dot{\zeta}_z = -\dot{\alpha}\ddot{\zeta}_z + \ddot{\phi}_z\dot{\zeta}_z + \dot{\phi}_z\ddot{\zeta}_z + \dot{\zeta}_z\dot{\phi}_{zr}\dot{\zeta} - \ddot{\phi}_{rr}\dot{\zeta} - \dot{\phi}_{rr}\ddot{\zeta} - \tfrac{1}{2}\dot{\phi}_{rrr}\dot{\zeta}^2, \quad \text{on} \quad r = 1, \tag{29}$$

$$\begin{aligned}\dddot{\zeta}_{zz} + \dddot{\zeta} + \alpha_0\dddot{\phi}_z + \ddot{\alpha}\dot{\phi}_z = &-\alpha_0(\dot{\phi}_{zr}\ddot{\zeta} + \ddot{\phi}_{zr}\dot{\zeta} + \tfrac{1}{2}\dot{\phi}_{zrr}\dot{\zeta}^2) - \dot{\alpha}(\ddot{\phi}_z + \dot{\phi}_{zr}\dot{\zeta}) \\ &+ \dot{\phi}_{rr}\dot{\phi}_r\dot{\zeta} + \ddot{\phi}_r\dot{\phi}_r + \dot{\phi}_{zr}\dot{\phi}_z\dot{\zeta} + \dot{\phi}_z\ddot{\phi}_z + 2\dot{\zeta}\ddot{\zeta} - \dot{\zeta}^3 + \tfrac{1}{2}\dot{\zeta}\dot{\zeta}_z^2 + \tfrac{3}{2}\dot{\zeta}_z^2\dot{\zeta}_{zz} - \dot{\zeta}_z\ddot{\zeta}_z, \quad \text{on} \quad r = 1.\end{aligned} \tag{30}$$

We note that the free surface boundary conditions have been expanded about $r = 1$. Equations (19)–(22), (23)–(26) and (27)–(30) constitute the first-, second- and third-order problems respectively. It is possible, of course, to carry the expansions further, but, to obtain the first non-trivial term of the finite amplitude effect on the Weber number or the propagation speed of a disturbance, the above expansions are sufficient.

Perturbation solutions

A solution which is periodic in the z-direction of (19) may be taken in the following form:

$$\dot{\phi} = AI_0(kr)\cos kz, \tag{31}$$

where A is a constant. The elimination of $\dot{\zeta}$ between (21) and (22) and the use of (31) give us:

$$\alpha_0(k) = \left[\left(k-\frac{1}{k}\right)\frac{I_1(k)}{I_0(k)}\right]^{\frac{1}{2}}. \tag{32}$$

From (31) and (21) we obtain

$$\dot{\zeta} = -\frac{AI_1(k)}{\alpha_0(k)}\sin kz. \tag{33}$$

We note that the above result satisfies equation (20).

From the obtained first-order solutions and the elimination of $\ddot{\zeta}$ between (25) and (26), we have as the boundary condition for $\ddot{\phi}$

$$\ddot{\phi}_{rzz}+\ddot{\phi}_r = \alpha_0^2(k)\ddot{\phi}_{zz}-2\dot{\alpha}(k)\alpha_0(k)Ak^2I_0(k)\cos kz$$
$$+\frac{3A^2k^2I_1^2(k)}{2\alpha_0(k)}\left[(1-3k^2)\frac{I_0(k)}{I_1(k)}+(k^2-1)\frac{I_1(k)}{I_0(k)}+k-\frac{1}{k}\right]\sin 2kz, \quad \text{on} \quad r=1. \tag{34}$$

If the coefficient of $\cos kz$ in (34) does not vanish we cannot find a periodic potential function $\ddot{\phi}$ satisfying equation (34). Therefore it is necessary that

$$\dot{\alpha}(k) = 0. \tag{35}$$

A solution of equation (23) which satisfies equation (34) with $\dot{\alpha}(k) = 0$ can easily be found to be of the form

$$\ddot{\phi} = \frac{A^2k^2I_1^2(k)}{\alpha_0(k)}BI_0(2kr)\sin 2kz, \tag{36}$$

where B is a constant. We could, of course, add a term of the form given in (31) but with an arbitrary constant multiplier. However, such a solution is discarded since we wish to allow only the first-order term of this form. This is equivalent to saying that we shall re-define the small parameter ϵ so as to make such a term disappear from the higher-order solutions. Substitution of (36) into (34) with $\dot{\alpha}(k) = 0$ yields

$$B = \frac{3\left[\dfrac{I_0(k)}{I_1(k)}(1-3k^2)+\dfrac{I_1(k)}{I_0(k)}(k^2-1)+k-\dfrac{1}{k}\right]}{8k^2I_0(2k)[\alpha_0^2(k)-\alpha_0^2(2k)]}, \tag{37}$$

where $\alpha_0(k)$ is given by (32). We note that for $k > 1$, $\alpha_0^2(k)$ is a monotonically increasing function of k, and therefore the denominator will never vanish. From (36), (25), (24) and the first-order solutions, we can obtain

$$\ddot{\zeta} = \frac{A^2k^2I_1^2(k)}{\alpha_0^2(k)}\left\{\left[BI_1(2k)+\frac{1}{4k^2}-\frac{I_0(k)}{2kI_1(k)}\right]\cos 2kz-\frac{1}{4k^2}\right\}. \tag{38}$$

This completes the second-order solutions.

Using a process similar to the one used above, the boundary condition for $\ddot{\phi}$ can be found and is of the form

$$\ddot{\phi}_{rzz}+\ddot{\phi}_r = \alpha_0^2(k)\ddot{\phi}_{zz}-\left[\frac{I_0(k)}{I_1(k)}\alpha_0(k)D_1+D_2\right]\alpha_0(k)k\cos kz+D_3\cos 3kz, \quad \text{on} \quad r=1, \tag{39}$$

where

$$D_1 = \ddot{\alpha}\frac{AkI_1(k)}{\alpha_0(k)}+\frac{A^3k^3I_1^3(k)}{\alpha_0^2(k)}\left\{\left[kI_0(2k)+\frac{1}{2}\left(\frac{kI_0(k)}{I_1(k)}-1\right)I_1(2k)\right]B + \frac{1}{4}\left[\frac{I_0(k)}{kI_1(k)}-\frac{3}{2k^2}-\frac{I_0^2(k)}{I_1^2(k)}-\frac{3}{2}\right]\right\}, \tag{40}$$

$$D_2 = \ddot{\alpha}AkI_0(k)+\frac{A^3k^3I_1^3(k)}{\alpha_0(k)}\left\{\left[k\frac{I_0(k)}{I_1(k)}I_0(2k)-\left(\tfrac{3}{2}k+\frac{I_0(k)}{I_1(k)}\right)I_1(2k)\right]B + \frac{I_0^2(k)}{2I_1^2(k)k}+\frac{3I_0(k)}{2k^2(k^2-1)I_1(k)}-\frac{1}{2k}\right\}. \tag{41}$$

The explicit expression for D_3 will not be given here. The requirement that the coefficient of $\cos kz$ should vanish gives us the condition

$$\frac{I_0(k)}{I_1(k)}\alpha_0(k)D_1+D_2 = 0. \tag{42}$$

By use of (40) and (41), we have

$$\ddot{\alpha}(k) = \tfrac{1}{2}\alpha_0(k)h(k)\zeta_0^2(k), \tag{43}$$

where $\zeta_0(k) = AI_1(k)/\alpha_0(k)$, which is the amplitude of the first-order surface disturbance, and

$$h(k) = \tfrac{1}{2}k^2\left\{\left[\left(3+3k\frac{I_1(k)}{I_0(k)}-\frac{kI_0(k)}{I_1(k)}\right)I_1(2k)-4kI_0(2k)\right]B + \frac{3}{4}-\frac{3I_0(k)}{2kI_1(k)}+\frac{3}{4k^2}+\frac{I_1(k)}{kI_0(k)}+\frac{I_0^2(k)}{2I_1^2(k)}-\frac{3}{k^2(k^2-1)}\right\}. \tag{44}$$

From the expression of $\ddot{\alpha}(k)$ it is noted that this perturbation method fails as k tends to unity.

Equation (16) may now be written as

$$\begin{aligned} W_b &= [\alpha_0(k)+\epsilon\dot{\alpha}(k)+\epsilon^2\ddot{\alpha}(k)+O(\epsilon^3)]^2 \\ &= \alpha_0^2(k)+2\epsilon^2\alpha_0(k)\ddot{\alpha}(k)+O(\epsilon^3) \\ &= \alpha_0^2(k)[1+(\epsilon\zeta_0)^2h(k)]+O(\epsilon^3). \end{aligned} \tag{45}$$

For given W_b, the above equation can give us the wave-number of the disturbance which appears to be stationary. To obtain it, let us expand k in the form

$$k = k^*+(\epsilon\zeta_0)\dot{k}^*+(\epsilon\zeta_0)^2\ddot{k}^*+O(\epsilon^3), \tag{46}$$

where k^* is the solution of the equation

$$W_b = \alpha_0^2(k^*), \tag{47}$$

which has been plotted in figure 2. Substitution of (46) into (45) yields

$$\left.\begin{aligned} &\dot{k}^* = 0 \\ \text{and} \quad &\ddot{k}^* = -\frac{W_b h(k^*)}{\left[\frac{d}{dk}\alpha_0^2(k)\right]_{k=k^*}} = -\frac{W_b I_0^2(k^*) h(k^*)}{[I_0^2(k^*) - I_1^2(k^*)]\left(k^* - \frac{1}{k^*}\right) + \frac{2}{k^{*2}} I_0(k^*) I_1(k^*)}. \end{aligned}\right\} \quad (48)$$

Hence, to an observer having a velocity defined by W_b relative to the jet, a disturbance on the jet appears to be stationary if the wave-number of the disturbance is

$$k = k^* - (\epsilon\zeta_0)^2 \frac{W_b I_0^2(k^*) h(k^*)}{[I_0^2(k^*) - I_1^2(k^*)]\left(k^* - \frac{1}{k^*}\right) + \frac{2}{k^{*2}} I_0(k^*) I_1(k^*)} + O(\epsilon^3), \quad (49)$$

where k^* is the solution of (47) and $h(k)$ is given by (44). Some numerical results of (49) are plotted in figure 3. In that figure we note that for a given Weber number the finite amplitude effect tends to shorten the wavelength of the stationary state.

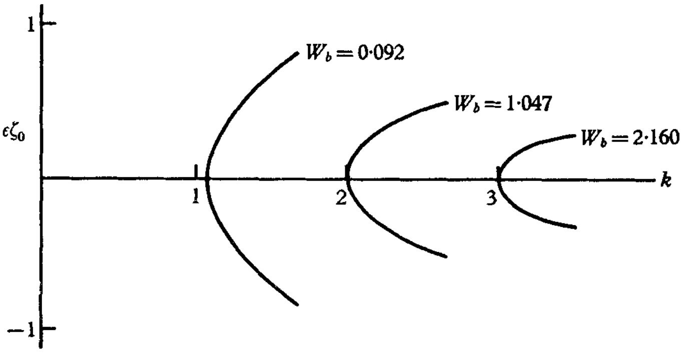

FIGURE 3. Finite amplitude effect on a stationary-state solution corresponding to a given Weber number.

4. Finite amplitude effect on the unstable states

The linearized solution given by (7) indicates that small disturbances with $m = 0$ and $k < 1$ may be unstable and, if they are unstable, their amplitude will increase exponentially fast. To study the non-linear effect on such types of transient states a formal perturbation expansion analogous to that used in the last section will be employed.

The free surface boundary conditions for this problem may again be obtained from (4) and (5). For axisymmetrical disturbances ($m = 0$), the free surface boundary conditions are

$$\left.\begin{aligned} &\phi_r = \phi_z \zeta_z + \zeta_t, \quad \text{on} \quad r = 1 + \zeta, \\ &\phi_t + \tfrac{1}{2}(\phi_r^2 + \phi_z^2) + \frac{1}{(1+\zeta_z^2)^{\frac{3}{2}}}\left(\frac{1+\zeta_z^2}{1+\zeta} - \zeta_{zz}\right) = 1, \quad \text{on} \quad r = 1 + \zeta. \end{aligned}\right\} \quad (50)$$

Let us now make the transformation

$$\tau = \sigma t, \tag{51}$$

where σ is a parameter, independent of r, z and t, which characterizes the growth rate of the unstable states. This transformation makes equations (50) become

$$\phi_r = \phi_z \zeta_z + \sigma \zeta_\tau, \quad \text{on} \quad r = 1 + \zeta(z, \tau), \tag{52}$$

and
$$\sigma\phi_\tau + \tfrac{1}{2}(\phi_r^2 + \phi_z^2) + \frac{1}{(1+\zeta_z^2)^{\frac{3}{2}}}\left(\frac{1+\zeta_z^2}{1+\zeta} - \zeta_{zz}\right) = 1, \quad \text{on} \quad r = 1 + \zeta(z, \tau). \tag{53}$$

We note that the forms of (52) and (53) are identical, respectively, with those of (12) and (13) if the operator $-W_0^{\frac{1}{2}}\partial/\partial z$ in (12) and (13) is replaced by $\sigma\,\partial/\partial\tau$. It is further noted that (14) and (15) are still valid except that now the perturbation potential ϕ and the free surface displacement ζ are dependent upon the variable τ.

Let us now assume that ϕ, ζ and σ may all be expanded in power series of a small parameter ϵ characterizing the initial amplitude of a small disturbance, or

$$\phi(r, z, \tau) = \epsilon\dot{\phi}(r, z, \tau) + \epsilon^2\ddot{\phi}(r, z, \tau) + \epsilon^3\dddot{\phi}(r, z, \tau) + O(\epsilon^4), \tag{54}$$

$$\zeta(z, \tau) = \epsilon\dot{\zeta}(z, \tau) + \epsilon^2\ddot{\zeta}(z, \tau) + \epsilon^3\dddot{\zeta}(z, \tau) + O(\epsilon^4), \tag{55}$$

$$\sigma = \sigma_0(k) + \epsilon\dot{\sigma}(k) + \epsilon^2\ddot{\sigma}(k) + O(\epsilon^3). \tag{56}$$

Similar to the processes used in the last section, the substitution of the perturbation expansions (54), (55) and (56) into the governing equations (14), (15), (52) and (53) will yield sets of equations which must be satisfied by $\dot{\phi}$, $\ddot{\phi}$, $\dddot{\phi}$, $\dot{\zeta}$, $\ddot{\zeta}$, $\dddot{\zeta}$, σ_0, $\dot{\sigma}$ and $\ddot{\sigma}$. Owing to the similarities in the governing equations, boundary conditions and the perturbation expansions, these sets of equations can be obtained from equations (19) to (30) by simply replacing the operators $\alpha_0\,\partial/\partial z$, $\dot{\alpha}\,\partial/\partial z$ and $\ddot{\alpha}\,\partial/\partial z$ by $-\sigma_0\,\partial/\partial\tau$, $-\dot{\sigma}\,\partial/\partial\tau$ and $-\ddot{\sigma}\,\partial/\partial\tau$, respectively, in (19)–(30). However, for simplicity, they will not be written out explicitly; instead (19)–(30) will again be used except that whenever any one of them is mentioned it is understood, from this point onwards, that the above stated replacements have been made.

Perturbation solutions

From (21) and (22) the boundary condition for $\dot{\phi}$ can be obtained as

$$\sigma_0^2(k)\dot{\phi}_{\tau\tau} = \dot{\phi}_r + \dot{\phi}_{rzz}, \quad \text{on} \quad r = 1, \tag{57}$$

We shall now assume that an initial disturbance periodic in the z-direction may be represented, up to the first order in ϵ, by the following expression:

$$\dot{\phi} = A(\tau)\, I_0(kr)\cos kz, \tag{58}$$

where $A(\tau)$ is an arbitrary function of τ. It is obvious that (58) satisfies (19). Substitution of (58) into (57) gives

$$\sigma_0^2(k)\, A_{\tau\tau}(\tau) - \frac{k(1-k^2)\, I_1(k)}{I_0(k)}\, A(\tau) = 0. \tag{59}$$

Since an initial disturbance on a system with a free surface is usually characterized by both the initial velocity and the initial free surface displacement, therefore, to describe a general disturbance two independent modes are required, or we let

$$A(\tau) = a_1 e^{\tau} + a_2 e^{-\tau}, \tag{60}$$

where a_1 and a_2 are constant. By use of (60), (59) gives

$$\sigma_0^2(k) = \frac{k(1-k^2)\,I_1(k)}{I_0(k)}. \tag{61}$$

Equations (21) and (58) give us

$$\dot{\zeta}_\tau = \frac{1}{\sigma_0(k)}\dot{\phi}_r = \frac{A(\tau)\,kI_1(k)}{\sigma_0(k)}\cos kz, \tag{62}$$

where $\sigma_0(k)$ is defined by

$$\sigma_0(k) = \left[\frac{k(1-k^2)\,I_1(k)}{I_0(k)}\right]^{\frac{1}{2}}. \tag{63}$$

Integrating (62) with respect to τ, we obtain the corresponding free surface displacement

$$\dot{\zeta} = \frac{\bar{A}(\tau)\,kI_1(k)}{\sigma_0(k)}\cos kz + f(z), \tag{64}$$

where $\bar{A}(\tau) = a_1 e^{\tau} - a_2 e^{-\tau}$ and $f(z)$ is an arbitrary periodic function of z with period equal to λ. In order to satisfy the dynamic boundary condition (22), $f(z)$ can only be zero. However, we recall that in formulating the problem a function of t has been deleted from the dynamic free surface boundary condition (5). This suggests that so far as a function of τ is concerned the balance of the dynamic boundary condition may be ignored. For this reason we may regard $f(z)$ to be an arbitrary constant instead of zero. Therefore we may write

$$\dot{\zeta} = \frac{\bar{A}(\tau)\,kI_1(k)}{\sigma_0(k)}\cos kz + \dot{c}, \tag{65}$$

where $\dot{c}$ is a constant. Substitution of (65) into (20) gives

$$\dot{c} = 0. \tag{66}$$

To obtain the boundary condition for $\ddot{\phi}$, we eliminate $\ddot{\zeta}$ between (25) and (26) and substitute the known first-order solutions into the resulting equation. This yields

$$\ddot{\phi}_{rzz} + \ddot{\phi}_r = \sigma_0^2(k)\,\ddot{\phi}_{\tau\tau} + 2\dot{\sigma}(k)\,\sigma_0(k)\,I_0(k)\,A(\tau)\cos kz + \frac{N(k)}{\sigma_0(k)}A(\tau)\bar{A}(\tau)\cos 2kz, \quad \text{on} \quad r = 1, \tag{67}$$

where

$$N(k) = \tfrac{3}{2}k^2 I_1^2(k)\left\{(1-k^2)\left[1 + k\frac{I_1(k)}{I_0(k)}\right] - k(1-3k^2)\frac{I_0(k)}{I_1(k)}\right\}. \tag{68}$$

Because the dynamic boundary condition may be regarded as indeterminate up to a function of τ at this stage, a function of τ has been deleted from (67). The requirement that the coefficient of $A(\tau)\cos kz$ in (67) must vanish gives us

$$\dot{\sigma}(k) = 0. \tag{69}$$

A solution of (23) which satisfies (67) with $\dot{\sigma}(k) = 0$ can be found to be of the form

$$\ddot{\phi} = B(\tau) I_0(2kr) \cos 2kz, \tag{70}$$

where $B(\tau)$ is an unknown function of τ. It should be mentioned that a term of the form of $A(\tau) I_0(kr) \cos kz$ can be added to the solution $\ddot{\phi}$ given in (70). However, for convenience, we shall assume that the small parameter ϵ has been so chosen that such a term is certain not to appear in the higher-order solutions. Substitution of (70) into (67) with $\dot{\sigma}(k) = 0$ yields

$$\sigma_0^2(k) B_{\tau\tau}(\tau) - \sigma_0^2(2k) B(\tau) = -\frac{N(k)}{\sigma_0(k) I_0(2k)} A(\tau) \bar{A}(\tau), \tag{71}$$

where $\sigma_0^2(k)$ is given by (61). A general solution of (71) may be written as

$$B(\tau) = \frac{N(k) A(\tau) \bar{A}(\tau)}{\sigma_0(k) I_0(2k) [\sigma_0^2(2k) - 4\sigma_0^2(k)]} + b_1 \exp\left[\frac{\sigma_0(2k)}{\sigma_0(k)} \tau\right] + b_2 \exp\left[-\frac{\sigma_0(2k)}{\sigma_0(k)} \tau\right], \tag{72}$$

where b_1 and b_2 are arbitrary constants, and

$$\sigma_0(2k) = \left[\frac{2k(1-4k^2) I_1(2k)}{I_0(2k)}\right]^{\frac{1}{2}}, \tag{73}$$

which may be purely real or purely imaginary. We shall now assume that the initial disturbance is applied in such a way that

$$b_1 = b_2 = 0. \tag{74}$$

This assumption will not disturb the present study of the finite amplitude effect on the unstable modes. For, if $\sigma_0(2k)$ is purely real, the disturbance represented by b_1 and b_2 is completely similar to the one currently being studied, and, if $\sigma_0(2k)$ is purely imaginary, the disturbance due to b_1 and b_2 is stable. From (25) $\ddot{\zeta}$ can be found to be

$$\ddot{\zeta} = \frac{k^2 I_1^2(k)}{2\sigma_0^2(k)} M(k) [A^2(\tau) + A_0] \cos 2kz - \frac{k^2 I_1^2(k)}{4\sigma_0^2(k)} A^2(\tau) + \ddot{c}, \tag{75}$$

where A_0 and $\ddot{c}$ are constants, and

$$M(k) = \frac{2I_1(2k) N(k)}{k I_1^2(k) I_0(2k) [\sigma_0^2(2k) - 4\sigma_0^2(k)]} + k\frac{I_0(k)}{I_1(k)} - \tfrac{1}{2}. \tag{76}$$

Substituting the obtained $\ddot{\phi}$ and $\ddot{\zeta}$ into the dynamic boundary condition (26) and then equating the coefficients of the terms $\cos 2kz$, we obtain

$$A_0 = -\left\{\frac{[I_0^2(k) + I_1^2(k)]\sigma_0^2(k) + (2+k^2) I_1^2(k)}{I_1^2(k)(1-4k^2) M(k)} + 2\right\} a_1 a_2, \tag{77}$$

where a_1 and a_2 are defined in (60). The substitution of the obtained $\dot{\zeta}$ and $\ddot{\zeta}$ into (24) yields

$$\ddot{c} = \frac{k^2 I_1^2(k)}{4\sigma_0^2(k)} [A^2(\tau) - \bar{A}^2(\tau)] = \frac{k^2 I_1^2(k)}{\sigma_0^2(k)} a_1 a_2. \tag{78}$$

Hence
$$\ddot{\zeta} = \frac{k^2 I_1^2(k)}{2\sigma_0^2(k)} M(k) [A^2(\tau) + A_0] \cos 2kz - \frac{\bar{A}^2(\tau) k^2 I_1^2(k)}{4\sigma_0^2(k)}. \tag{79}$$

This completes the second-order solutions.

From the obtained first- and second-order solutions, and (29) and (30), the boundary condition for $\ddot{\phi}$ can be found and is in the form

$$\ddot{\phi}_{rzz}+\ddot{\phi}_r = \sigma_0^2(k)\ddot{\phi}_{\tau\tau}+[2\sigma_0(k)I_0(k)\ddot{\sigma}(k)-g(k)a_1a_2]A(\tau)\cos kz+E_1(k,\tau)\cos kz +E_2(k,\tau)\cos 3kz, \quad (80)$$

where

$$g(k) = \frac{k^2}{4(1-4k^2)}\left\{3I_0(k)+\frac{k}{I_1(k)}[I_0^2(k)+I_1^2(k)]\right\}\{[I_0^2(k)+I_1^2(k)]\sigma_0^2(k)+(2+k^2)I_1^2(k)\}$$
$$-\frac{3}{4\sigma_0^2(k)}k^3I_1^3(k)(5+k^2)+\tfrac{11}{8}k^4I_1^2(k)I_0(k)-\tfrac{5}{4}k^3I_1^3(k)+k^3I_0^2(k)I_1(k)$$
$$-\tfrac{1}{4}k^4I_0^3(k)-\tfrac{9}{8}k^2I_1^2(k)I_0(k)-\frac{kN(k)}{I_0(2k)[\sigma_0^2(2k)-4\sigma_0^2(k)]}$$
$$\times\left\{2kI_0(k)I_0(2k)+\left[\frac{kI_0^2(k)}{2I_1(k)}-\tfrac{3}{2}I_0(k)-\tfrac{3}{2}kI_1(k)\right]I_1(2k)\right\}. \quad (81)$$

The explicit expressions for $E_1(k,\tau)$ and $E_2(k,\tau)$ will not be given here. Again, the requirement that the coefficient of $A(\tau)\cos kz$ in (80) should vanish yields the condition

$$2\sigma_0(k)I_0(k)\ddot{\sigma}(k)-g(k)a_1a_2 = 0, \quad (82)$$

which in turn gives

$$\ddot{\sigma}(k) = \frac{g(k)}{2\sigma_0(k)I_0(k)}a_1a_2. \quad (83)$$

From the expressions for $g(k)$ and $\sigma_0(k)$, we note that as k tends to 0·5 or 1·0 from inside the interval (0·5, 1·0) the quantity $g(k)/2\sigma_0(k)I_0(k)$ in (83) tends to $-\infty$. Therefore this perturbation method fails when k tends to 0·5 or 1·0. By (56), (69) and (83), we have

$$\sigma(k) = \sigma_0(k)+\epsilon^2a_1a_2\frac{g(k)}{2\sigma_0(k)I_0(k)}+O(\epsilon^3). \quad (84)$$

From the above results it is interesting to note that, if the initial disturbance is so adjusted that it yields no decaying mode whose initial amplitude is represented by $a_2 = 0$, then up to the second-order term in ϵ the growth rate of such a disturbance depends only on the wave-number k in exactly the same way as has been demonstrated by the linearized theory. However, for general initial disturbances both the subsequent growth rate and the decaying rate of the disturbances are affected by the magnitudes of the initial disturbances.

We shall now regard the amplitudes of the initial disturbances ϵa_1 and ϵa_2 as small given quantities and try to find the fastest-growing mode. From the linearized theory it is seen that the fastest-growing mode, or the maximum value of $\sigma_0(k)$, occurs at $k = k^+ = 0{\cdot}697$. For ϵa_1 and ϵa_2 sufficiently small it is conceivable that a maximum value of $\sigma(k)$ should exist in the neighbourhood of k^+. To find this maximum value we simply differentiate $\sigma(k)$ given by (84) with respect to k and set the resulting equation equal to zero, or

$$\frac{d\sigma(k)}{dk} = \frac{d\sigma_0(k)}{dk}+\epsilon^2a_1a_2\frac{d}{dk}\left[\frac{g_1(k)}{2\sigma_0(k)I_0(k)}\right]+O(\epsilon^3) = 0. \quad (85)$$

To solve the above equation we may write the solution for k as

$$k = k^+ + \epsilon\dot{k}^+ + \epsilon^2 a_1 a_2 \ddot{k}^+ + O(\epsilon^3), \tag{86}$$

where $k^+ = 0{\cdot}697$ is the solution of $d\sigma_0(k^+)/dk^+ = 0$. Substituting (86) into (85) and then differentiating the resulting equation with respect to ϵ, we have

$$[\dot{k}^+ + 2\epsilon a_1 a_2 \ddot{k}^+ + O(\epsilon^2)]\frac{d^2\sigma_0(k)}{dk^2} + 2\epsilon a_1 a_2 \frac{d}{dk}\left[\frac{g_1(k)}{2\sigma_0(k)I_0(k)}\right] + O(\epsilon^2) = 0. \tag{87}$$

Setting ϵ in (87) equal to zero and since $d^2\sigma_0(k^+)/dk^{+2} < 0$, we obtain

$$\dot{k}^+ = 0. \tag{88}$$

Similarly, differentiating (87) with respect to ϵ and then setting ϵ in the resulting equation to zero, we have

$$\ddot{k}^+ = -\frac{\left\{\dfrac{d}{dk}\left[\dfrac{g_1(k)}{2\sigma_0(k)I_0(k)}\right]\right\}_{k=k^+}}{\left(\dfrac{d^2\sigma_0(k)}{dk^2}\right)_{k=k^+}}. \tag{89}$$

After carrying out the differentiations and substitution of $k^+ = 0{\cdot}697$ in (89), $\ddot{k}^+$ becomes

$$\ddot{k}^+ = -3{\cdot}255. \tag{90}$$

For ϵa_1 and ϵa_2 sufficiently small the most unstable mode is therefore given by

$$k = 0{\cdot}697 - 3{\cdot}255\epsilon^2 a_1 a_2. \tag{91}$$

The above result clearly indicates that the most unstable wavelength is a function of the magnitudes of the initially applied disturbances. This may help to explain the experimentally observed fact that the most unstable wavelength usually varies over a certain range, instead of being a constant (Haenlein 1932).

Since the solution fails to be valid at $k = 1{\cdot}0$ and $0{\cdot}5$, and possibly at some more points having the value of k less than $0{\cdot}5$ when higher-order terms are sought, it seems that the present perturbation method in treating unstable states is somewhat limited. However, when only those states in the neighbourhood of the most unstable state are concerned, this perturbation method does give us some analytical insight into the problem of the finite amplitude effect. On the other hand, if one is interested in achieving a uniformly valid perturbation method for the unstable case one may try to formulate the problem by using a method analogous to those methods used in the theory of long waves in shallow water. The analogy between these two problems lies in the fact that a small, periodic, axisymmetric disturbance applied to a jet of circular cross-section may be unstable only if the wavelength of the disturbance is at least 2π times the radius of the jet. This in a way is very similar to the problem of long waves in shallow water. A formulation along this line has been given by Moiseev (1965).

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REFERENCES

CONCUS, P. 1962 Standing capillary–gravity waves of finite amplitude. *J. Fluid Mech.* **14**, 568–76.

GRANT, R. P. & MIDDLEMAN, S. 1966 Newtonian jet stability. *A.I.Ch.E. J.* **12**, 669–78.

HAENLEIN, A. 1932 Distintegration of a liquid jet. *NACA Tech. Memo.* no. 659.

KELLER, J. B. & TING, L. 1966 Periodic vibrations of systems governed by nonlinear partial differential equations. *Comm. Pure Appl. Math.* **19**, 371–420.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

MOISEEV, N. N. 1965 On some properties of fluid flow under the action of surface tension forces. *PMM* **29**, 1015–22. English translation: 1966 *Appl. Math. Mech.* **29**, 1197–1205.

RAYLEIGH, LORD 1878 On the instability of jets. *Proc. Lond. Math. Soc.* **10**, 4–13.

RAYLEIGH, LORD 1879 On the capillary phenomena of jets. *Proc. Roy. Soc.* **29**, 71–97.

SAVART, F. 1833 Mémoire sur la constitution des veines liquides lancés par des orifices circulaires en mince paroi. *Ann. Chim.* LIII, 337–86.

SEGEL, L. A. 1965 Non-linear hydrodynamic stability theory and its applications to thermal convection and curved flows. *Non-equilibrium Thermodynamics*, pp. 165–97. University of Chicago Press.

TADJBAKHSH, I. & KELLER, J. B. 1960 Standing surface waves of finite amplitude. *J. Fluid Mech.* **8**, 442–51.

VERMA, G. R. & KELLER, J. B. 1962 Three-dimensional standing surface waves of finite amplitude. *Physics Fluids*, **5**, 52–6.

WEBER, C. 1931 Zum Zerfall eines Flüssigkeitsstrahles. *ZAMM*, **11**, 136–54.

WEHAUSEN, J. V. & LAITONE, E. V. 1960 *Surface Waves. Encyclopedia of Physics*, IX, 653–67. Berlin: Springer-Verlag.